M.Sc. Sem II, Paper VI

Complex Analysis (Mobius Transformations)

A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called bilinear or linear fractional transformation where $a, b, c, d \in$ and A bilinear transformation $S(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is called Mobius map or Mobius transformation.

(1) Mobius transformation is one-one and onto

(2) If $S(z) = \frac{az+b}{cz+d}$, then $S^{-1}(w) = \frac{-dw+b}{cw-a}$.

(3) If S and T are Mobius transformations then $S \circ T$ is also Mobius transfomation.

(4) $S(z) = z + a$ (Translation)

$S(z) = az$ (Dilation/Magnification)

$S(z) = e^{i\theta} z$ (Rotation)

$S(z) = \frac{1}{z}$ (Inversion)

Theorem:- If S is a Mobius transformation then S is composition of translation, dilation and inversion.

Proof- Let $S(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ be Mobius transformation.

Case 1. When $c = 0$ then $S(z) = \left(\frac{a}{d}\right) z + \frac{b}{d}$

Let $S_1(z) = \left(\frac{a}{d}\right) z$, $S_2(z) = z + \frac{b}{d}$

Then $S_2 \circ S_1(z) = S_2(S_1(z)) = S_2\left(\left(\frac{a}{d}\right) z\right) = \left(\frac{a}{d}\right) z + \left(\frac{b}{d}\right) = S(z)$

→ Contd. on 2

Thus $S = S_2 \circ S_1$.

Case 2. When $c \neq 0$

let $S_1(z) = z + \frac{d}{c}$, $S_3(z) = \frac{bc-ad}{c^2} z$

$S_2(z) = \frac{1}{z}$, $S_4(z) = z + \frac{a}{c}$.

Then

$$S_4 \circ S_3 \circ S_2 \circ S_1(z) = S_4 \circ S_3 \circ S_2(S_1(z))$$
$$= S_4 \circ S_3 \circ S_2\left(z + \frac{d}{c}\right)$$
$$= S_4 \circ S_3\left[S_2\left(z + \frac{d}{c}\right)\right]$$
$$= S_4 \circ S_3\left(\frac{1}{z + \frac{d}{c}}\right)$$
$$= S_4\left[S_3\left(\frac{1}{z + \frac{d}{c}}\right)\right]$$
$$= S_4\left[\frac{bc-ad}{c^2}\left(\frac{1}{z + \frac{d}{c}}\right)\right]$$
$$= \left(\frac{bc-ad}{c(cz+d)}\right) + \frac{a}{c}$$
$$= \frac{az+b}{cz+d} = S(z).$$

Thus $S = S_4 \circ S_3 \circ S_2 \circ S_1$. (Proved)

Theorem:- Every Mobius transformation can have at most two fixed points.

Proof:- Let $S(z) = \frac{az+b}{cz+d}$ with $ad-bc \neq 0$ be Mobius transformation

Let z be fixed point of $S(z)$ then $S(z) = z$

$$\frac{az+b}{cz+d} = z \Rightarrow cz^2 + (d-a)z - b = 0$$

Which is quadratic in z. Hence it can have at most two roots. Therefore every Mobius transformation can have at most two fixed points otherwize $S(z) = z$ for all z (Identity map).